BERTRAND CURVES

BY

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CONTENTS

BERTRAND CURVES

Bertrand proposed the problem: To determine the curves whose principal normals are the principal normals of another curve. If a curve fulfilling these conditions exists, it is known as a Bertrand Curve. The relation is a reciprocal one, i.e., if one curve is a Bertrand of a second, then the second is a Bertrand of the first.

It is the purpose of this article to determine the condition for the existence of a Bertrand Curve from two standpoints:

First, with the curve given in terms of a parameter and referred to stationary axes; and,

Second, with the curve referred to the moving trihedral;

to show some examples of these curves; and, finally, to examine certain curves, defined in algebraic functions of a parameter, and find whether these curves satisfy the conditions for Bertrand curves.

I

BERTRAND CURVES

FROM THE ORDINARY NOTIONS OF ANALYTIC GEOMETRY

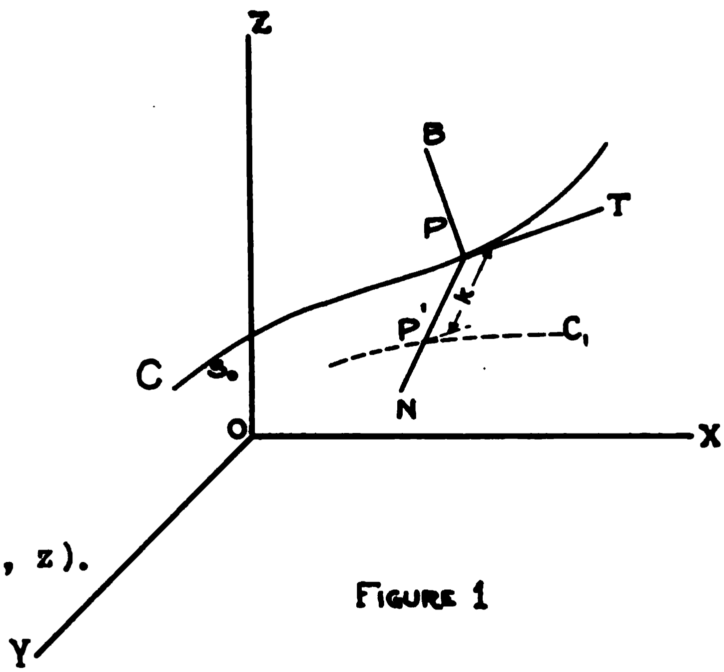

FIGURE 1

Let the curve C be defined in terms of the arc, s, measured from some point s_o (Fig. 1). Let the lines PT, PN, and PB be the tangent, normal, and binormal, respectively, to C at the point $P(x, y, z)$.

If the curve C has a Bertrand Curve there will be some point P' on PN at a distance K from P, which will be a point of the Bertrand Curve. If the direction cosines of PN, with respect to the axes at O, are ℓ, m, and n, the coördinates of the point P' (ξ, η, ζ) will be $\xi = x + K\ell$, $\eta = y + Km$, $\zeta = z + Kn$. The direction cosines of the tangent to the curve C_1 at P' are

$$\frac{d\xi}{ds_1} = \frac{d\xi}{ds} \cdot \frac{ds}{ds_1} = (x' + K\ell' + \ell K')\frac{ds}{ds_1},$$

Similarly, where s_1 is the arc of the curve C_1,

$$\frac{d\eta}{ds_1} = (y' + Km' + mK')\frac{ds}{ds_1},$$

and

$$\frac{d\zeta}{ds_1} = (z' + Kn' + nK')\frac{ds}{ds_1}.$$

The normal and tangent to the curve C_1 at point P' are at right angles.

Since the curves C and C_1 have a common normal, the condition for a right angle is,

$$\ell(x' + K\ell' + \ell K') + m(y' + Km' + mK') + n(z' + Kn' + nK') = 0,$$

or

$$\ell x' + my' + nz' + K(\ell\ell' + mm' + nn') + K'(\ell^2 + m^2 + n^2) = 0.$$

Since the normal and the tangent to C are perpendicular

$$\ell x' + my' + nz' = 0;$$

Also by differentiation of

$$\ell^2 + m^2 + n^2 = 1,$$

$$\ell\ell' + mm' + nn' = 0.$$

Therefore $$K' = 0.$$

Consequently, K is a constant.

It follows, therefore, that one condition for a Bertrand Curve is that it must intercept the normals of the given curve at a constant distance from the points of the curve.

The direction cosines of the binormal to the curve C_1 at P' are proportional to $(\eta'\zeta'' - \eta''\zeta')$, $(\zeta'\xi'' - \zeta''\xi')$, and $(\xi'\eta'' - \xi''\eta')$. By differentiation of the expressions for the direction cosines of the tangent to the curve C_1 at P', keeping in mind that K is constant, and expressing the fact that

the tangent and binormal are perpendicular, the following is obtained:

$$
\begin{aligned}
&\ell[(y'+Km')(z''+Kn'') - (y''+Km'')(z'+Kn')] + \\
(1)\qquad &m[(z'+Kn')(x''+K\ell'') - (z''+Kn'')(x'+K\ell')] + \\
&n[(x'+K\ell')(y''+Km'') - (x''+K\ell')(y'+Km')] = 0.
\end{aligned}
$$

The substitution of the values obtained from the Frenet–Serret formulas, and from differentiation of these formulas, reduces (1) to the form

(2)[1] $$\frac{\tau'}{\tau^2} + \frac{\tau' K}{\rho\tau^2} + \frac{\rho' K}{\rho^2\tau^2} = 0,$$

where $\frac{1}{\rho}$ and $\frac{1}{\tau}$ are the curvature and torsion, respectively, of the curve C at P.

Equation (2) may be put in the form

(3) $$\frac{1}{\tau}\, d\left(1 - \frac{K}{\rho}\right) = \left(1 - \frac{K}{\rho}\right) d\left(\frac{1}{\tau}\right).$$

The solution of (3) gives

(4) $$\log \frac{1}{\tau} = \log\left(1 + \frac{K}{\rho}\right) + C.$$

Putting the constant as log c and on the left side of the equation, (4) reduces to

(5) $$\frac{c}{\tau} + \frac{K}{\rho} = 1$$

where c is a constant of integration.

Equation (5) establishes the second necessary condition for a Bertrand curve; if a curve has a Bertrand curve, there

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[1]The complete reduction of (1), which is rather long, is given as Appendix A to this article.

must be a linear relation between the curvature and torsion of the given curve. The general linear equation between $\frac{1}{T}$ and $\frac{1}{\rho}$ is

$$\frac{\ell}{T} + \frac{M}{\rho} = N;$$

If such a relation can be established for a curve, c in (5) is $\frac{\ell}{N}$, and K is $\frac{M}{N}$. The problem of determining c and K is thus the same as establishing a general linear relation where ℓ, M, and N are constants.

II

THE MOVING TRIHEDRAL

If a curve is referred to the tangent, normal, and binormal as axes, the coördinates at a point $s_0 = 0$ are

(1)[1]
$$x = s - \frac{1}{6\rho^2} s^3 + \dots,$$
$$y = \frac{s^2}{2\rho} - \frac{1}{6}\frac{\rho'}{\rho^2} s^3 + \dots,$$
$$z = \frac{1}{6\rho\tau} s^3 + \dots,$$

where $\frac{1}{\rho}$ and $\frac{1}{\tau}$ are the curvature and torsion at the point $s_0 = 0$, and the unwritten terms are of the fourth or higher order in s.

As the point from which s_0 is measured is not determined, there may be a set of axes for every point of the curve. In other words, a point moving along the curve may be considered as the intersection of three mutually perpendicular lines, which keep rotating so as always to coincide with the tangent, normal, and binormal of the given curve. The configuration, with the idea of motion in mind, is called the moving trihedral.

With reference to the trihedral at any point, the direction cosines of the tangent, normal, and binormal have the values

***

[1] EISENHART, Differential Geometry, p. 18.

$$(2) \qquad \begin{array}{lll} \alpha = 1, & \beta = 0, & \gamma = 0; \\ \ell = 0, & m = 1, & n = 0; \\ \lambda = 0, & \mu = 0, & \nu = 1. \end{array}$$

As the trihedral begins to move, the rates of change of these functions with respect to s are found from the Frenet formulas to be

$$(3) \qquad \begin{array}{lll} \frac{d\alpha}{ds} = 0, & \frac{d\beta}{ds} = \frac{1}{\rho}, & \frac{d\gamma}{ds} = 0; \\ \frac{d\ell}{ds} = -\frac{1}{\rho}, & \frac{dm}{ds} = 0, & \frac{dn}{ds} = -\frac{1}{\tau}; \\ \frac{d\lambda}{ds} = 0, & \frac{d\mu}{ds} = \frac{1}{\tau} & \frac{d\nu}{ds} = 0. \end{array}$$

If $P_0(x, y, z)$ and $P_1(x_1, y_1, z_1,)$ are two points on a curve, Δs the length of the arc PP_1, the cosines of the angles between the two sets of axes may be found from (3). For example, the rate of change of α is 0, and from (2) $\alpha = 1$ at P_0, then the cosine of the angle between the x and x_1 axes is 1 to within terms of an order higher than the first in Δs; then the cosine of the angle between the x_1 and the y axes is $\frac{\Delta s}{\rho}$, to within higher powers of Δs. The cosines of the other seven angles may be found in a like manner.

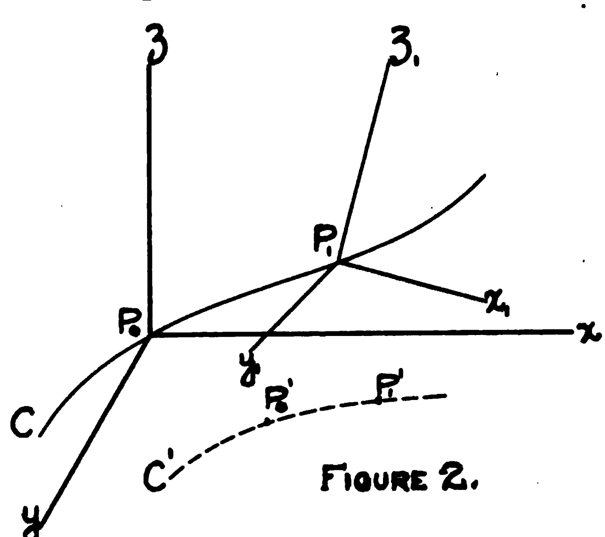

FIGURE 2.

Let P_0' be a point whose coördinates with respect to the axes at P_0 are ξ, η, ζ, and let P_0' trace a curve C' as

P_0 describes the curve C. It may happen that P_0' will be fixed, relative to the moving trihedral, but ordinarily the change in the coördinates of P_0' will be due, not only to the motion of the trihedral, but also to the motion of P_0' relative to the trihedral, as P_0' moves along C'.

From (1) the coördinates of P_1 may be taken as Δs, 0, 0, neglecting terms of order higher than the first in Δs. The axes at P_1 have been subject to a translation in the direction of the tangent, and to a rotation which can be measured, due to the fact, that the direction cosines of the angles between the new and the old axes have been found.

If P_1' represents a point on the curve C' corresponding to the point P_1 on C the coordinates of the point P_1', relative to the axes at P_0 and P_1, may be written

$$\xi + \Delta_1\xi, \qquad \eta + \Delta_1\eta, \qquad \zeta + \Delta_1\zeta,$$

and

$$\xi + \Delta_2\xi, \qquad \eta + \Delta_2\eta, \qquad \zeta + \Delta_2\zeta,$$

in which Δ_1 indicates variation of a function with respect to the axes at P_0, and Δ_2 variation relative to the moving trihedral.

The application of the formulas for translation and rotation of axes gives

$$\begin{aligned}
\xi + \Delta_1\xi &= \Delta s + (\xi + \Delta_2\xi) - (\eta + \Delta_2\eta)\frac{\Delta s}{\rho}, \\
\eta + \Delta_1\xi &= \qquad (\xi + \Delta_2\xi)\frac{\Delta s}{\rho} + (\eta + \Delta_2\eta) + (\zeta + \Delta_2\zeta)\frac{\Delta s}{\tau}, \\
\zeta \quad \Delta_1\zeta & \qquad\qquad\qquad - (\eta + \Delta_2\eta)\frac{\Delta s}{\tau} + (\zeta + \Delta_2\zeta).
\end{aligned}$$

These reduce to

$$
(5)\qquad
\begin{aligned}
\frac{\Delta_1\xi}{\Delta s} &= \frac{\Delta_2\xi}{\Delta s} + 1 - \frac{\eta + \Delta_2\eta}{\rho} \\
\frac{\Delta_1\eta}{\Delta s} &= \frac{\Delta_2\eta}{\Delta s} + \frac{\xi + \Delta_2\xi}{\rho} + \frac{\zeta + \Delta_2\zeta}{\tau} \\
\frac{\Delta_1\zeta}{\Delta s} &= \frac{\Delta_2\zeta}{\Delta s} - \frac{\eta + \Delta_2\eta}{\tau}\,.
\end{aligned}
$$

The limit as P_1 approaches P_0 is

$$
(6)\qquad
\begin{aligned}
\frac{\delta\xi}{ds} &= \frac{d\xi}{ds} + 1 - \frac{\eta}{\rho} \\
\frac{\delta\eta}{ds} &= \frac{d\eta}{ds} + \frac{\xi}{\rho} + \frac{\zeta}{\tau} \\
\frac{\delta\zeta}{ds} &= \frac{d\zeta}{ds} - \frac{\eta}{\tau}\,,
\end{aligned}
$$

where δ represents the change of a function with respect to stationary axes, and d gives the change of the function relative to the moving trihedral.

If t denotes the distance between the two points P_0' and P_0' then $t^2 = (\xi - \xi_1)^2 + (\eta - \eta_1)^2 + (\zeta - \zeta_1)^2$ and (6) gives

(7)[1] $$\frac{dt}{ds} = \frac{\delta t}{ds}\,.$$

If a, b, c denote the direction cosines of the line P_0' P_1' with respect to the axes at P_0, then

$$\xi_1 = \xi + at, \qquad \eta_1 = \eta + bt, \qquad \zeta_1 = \zeta + ct.$$

If ξ_1, η_1, and ζ_1 satisfy (6)

(8)[1]
$$
\begin{aligned}
\frac{\delta a}{ds} &= \frac{da}{ds} - \frac{b}{\rho}\,, \\
\frac{\delta b}{ds} &= \frac{db}{ds} + \frac{a}{\rho} + \frac{c}{\tau} \\
\frac{\delta c}{ds} &= \frac{dc}{ds} - \frac{b}{\tau}
\end{aligned}
$$

***

[1]For complete reduction, see Appendix B.

If the line $P_0'P_1'$ remains fixed in direction $\frac{\delta a}{ds}$, $\frac{\delta b}{ds}$ and $\frac{\delta c}{ds}$ become zero, as there is no variation of the functions themselves but only variation due to the motion of the trihedral. In this case (8) reduces to

$$\frac{da}{ds} = \frac{b}{\rho},$$

(9)
$$\frac{db}{ds} = -\left(\frac{a}{\rho} + \frac{c}{\tau}\right)$$

$$\frac{dc}{ds} = \frac{b}{\tau}$$

In the same manner if P_0' is fixed in space, (6) becomes

$$\frac{d\xi}{ds} = \frac{\eta}{\rho} - 1,$$

(10)
$$\frac{d\eta}{ds} = -\frac{\xi}{\rho} + \frac{\zeta}{\tau}$$

$$\frac{d\zeta}{ds} = \frac{\eta}{\tau}.$$

III

APPLICATION OF THE MOVING TRIHEDRAL TO BERTRAND CURVES

If the curve C be referred to the moving trihedral as axes, the problem of finding the Bertrand curves of C is determining the conditions under which the point $P_1\ (0,\ K, 0)$ will generate a curve whose normal will coincide with the normal of the curve C; in other words, the two curves must have the same y axis when they are referred to the moving trihedral.

Since the point P_1 remains on the y axis

$$\frac{dx}{ds} = \frac{dz}{ds} = 0,$$

as there is no variation of x and z relative to the trihedral. And since P_1 tends to move at right angles to the y axis

$$\frac{\delta y}{ds} = 0.$$

For these reasons the equations in (6) reduce to

$$\frac{\delta x}{ds} = 1 - \frac{K}{\rho},$$

$$\frac{dK}{ds} = 0,$$

$$\frac{\delta z}{ds} = -\frac{K}{\tau}.$$

From

$$\frac{dK}{ds} = 0, \quad K = C, \tag{11}$$

that is, the Bertrand curve must cut each normal of the given

curve at a constant distance from the corresponding point on the given curve.

If ω is the angle between the tangents at P and P_1

$$\tan \omega = \frac{\delta z}{\delta x},$$

where z and x are co-ordinates of P_1, or

$$\tan \omega = \frac{-\frac{K}{\tau}}{1 - \frac{K}{\rho}} = \frac{K\rho}{\tau(K - \rho)}.$$

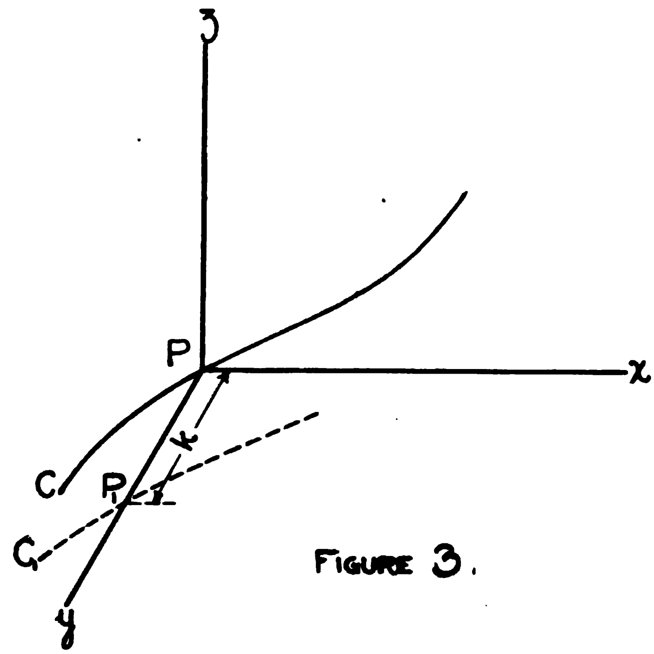

FIGURE 3.

Since the two curves have the same y axis the $x\,z$ planes are parallel. This relation is easily reduced to

$$\frac{\sin \omega}{\rho} - \frac{\cos \omega}{\tau} = \frac{\sin \omega}{K} \tag{12}$$

From

$$\frac{\delta s_1}{ds} = \left[\left[\frac{\delta x}{ds}\right]^2 + \left[\frac{\delta y}{ds}\right]^2 + \left[\frac{\delta z}{ds}\right]^2\right]^{\frac{1}{2}},$$

where s_1 is the arc of the curve C_1

$$\frac{\delta s_1}{ds} = \left[1 - \frac{K}{\rho}\right]^2 + \frac{K^2}{\tau^2},$$

$$= \left[- \frac{K}{\tan \omega}\right]^2 + \frac{K^2}{\tau^2} = \frac{K^2}{\sin^2 \omega}.$$

Therefore

$$\frac{\delta s_1}{ds} = \pm \frac{K}{\tau \sin \omega}. \tag{13}$$

Since the figures have been drawn for a dextrorsum curve, that is to say, one in which a point moving along the curve in the direction of increasing s passes from the negative to the

positive side of the osculating plane, it must be assumed that $\frac{1}{\tau}$ is negative. Consequently, the negative sign in (13) must be taken to make $\frac{\delta s_1}{ds}$ positive, and the two arcs must be counted in the same sense if the normals are to coincide.

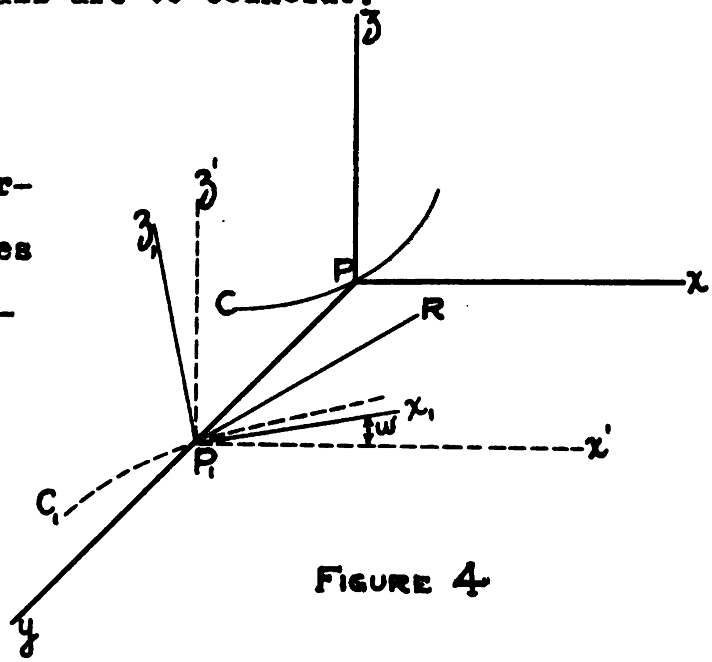

FIGURE 4

The curve C_1 (Fig. IV) will have a moving trihedral at point P_1. If it is a Bertrand curve of C the y axes of the two trihedrals will coincide.

If a_1, b_1, c_1 represent the direction cosines of a fixed line P_1R with reference to the trihedral at P_1 they will satisfy (9), that is,

$$
(14) \qquad
\begin{aligned}
\frac{da_1}{ds_1} &= \frac{b_1}{\rho_1}, \\
\frac{db_1}{ds_1} &= -\left[\frac{a_1}{\rho_1} + \frac{c_1}{\tau_1}\right], \\
\frac{dc_1}{ds_1} &= \frac{b_1}{\tau_1}.
\end{aligned}
$$

If a, b, c represent the direction cosines of the same line with respect to the trihedral at P

$$
\begin{aligned}
a_1 &= a \cos \omega + c \sin \omega \\
b_1 &= b, \\
c_1 &= a \sin \omega + c \cos \omega,
\end{aligned}
$$
[1]

where ω is the angle between the x and x_1 axes. When these values are substituted in (14)

or
$$\frac{d(a\cos\omega + c\sin\omega)}{ds_1} = \frac{b}{\rho_1}$$

$$\frac{d(a\cos\omega + c\sin\omega)}{ds}\cdot\frac{ds}{ds_1} = \frac{b}{\rho_1}.$$

By (13) this reduces to

$$\frac{b}{\rho}\cos\omega - a\sin\omega\frac{d\omega}{ds} + \frac{b}{\tau}\sin\omega + c\cos\omega\frac{d\omega}{ds} = -\frac{Kb}{\tau\rho_1\sin\omega}.$$

This may be grouped in the form

$$(15)\quad \left[\frac{\cos\omega}{\rho} + \frac{\sin\omega}{\tau} + \frac{K}{\tau\rho_1\sin\omega}\right]b + (c\cos\omega - a\sin\omega)\frac{d\omega}{ds} = 0.$$

In the same manner

$$(16)\quad \left[\frac{\tau\sin\omega}{K\rho} + \frac{\cos\omega}{\rho_1} - \frac{\sin\omega}{\tau_1}\right]a + \left[\frac{\sin\omega}{K} + \frac{\sin\omega}{\rho_1} + \frac{\cos\omega}{\tau_1}\right]c = 0,$$

and

$$(17)\quad \left[\frac{\sin\omega}{\rho} - \frac{\cos\omega}{\tau} - \frac{K}{\tau\tau_1\sin\omega}\right]b + (a\cos\omega + c\sin\omega)\frac{d\omega}{ds} = 0.$$

Equalities (15), (16) and (17) hold for every fixed line. They are therefore identities and consequently the coefficients of a, b, and c are zero.

From (15)
$$\cos\omega\frac{d\omega}{ds} = 0,$$
and
$$\sin\omega\frac{d\omega}{ds} = 0.$$

Since $\sin\omega$ and $\cos\omega$ cannot both equal zero,

$$(18)\quad \frac{d\omega}{ds} = 0.$$

Therefore, ω is constant.

From (17) and (12)

$$(19)\quad \tau_1\tau = \frac{K^2}{\sin^2\omega}.$$

From (16)

$$\frac{\sin\omega}{\rho_1} + \frac{\cos\omega}{\tau_1} + \frac{\sin\omega}{K} = 0. \tag{20}$$

Since by (18) ω is constant, equation (12) is a linear relation between the curvature and torsion of the given curve. Equation (20) shows that a similar relation holds between the curvature and torsion of the Bertrand curve. Conversely, given a curve C whose curvature and torsion satisfy the relation

$$\frac{A}{\rho} + \frac{B}{\tau} = C, \tag{21}$$

by giving K the value $\frac{A}{C}$, and ω the value $\tan^{-1} - \frac{A}{B}$, the curve whose intrinsic equations are found by substitution in (19) and (20) will satisfy the condition in (14) identically, and the point $0, K, 0$ will trace a Bertrand curve of C. Equation (21) is therefore a sufficient condition that a curve C should have a Bertrand Curve. The conclusion may be drawn: A necessary and sufficient condition that the normals of one curve be the normal of another is that a linear relation with constant coefficients exist between the curvature and torsion of the first curve.

It might be added that corresponding points on the two curves are a constant distance apart [from (11)], measured along the common normal; that the osculating planes cut under a constant angle [from (18)]; and that the torsions of the two curves have the same sign [from (19)].

IV

EXAMPLES OF BERTRAND CURVES

It has been found that there must be a linear relation between the curvature and torsion of a curve, if the given curve is to have a Bertrand curve.

If the relation is expressed

(1) $$\frac{A}{\rho} + \frac{B}{\tau} = C ,$$

A consideration of the equation will give some information in regard to special curves.

By Section II, the distance between the two curves is given by $K = \frac{A}{C}$, where A and C have the values in (1).
If $C = 0$, $A \neq 0$, $\frac{\rho}{\tau} = -\frac{A}{B}$, i.e., the ratio between the curvature and torsion is constant, and the curve is a helix.
If $K = \frac{A}{C} = \infty$, the Bertrand curve is at infinity.

If $A = 0$, $\frac{1}{\tau}$ is constant, and $K = \frac{A}{C} = 0$.
From this, if a curve has constant torsion the Bertrand curve coincides with the given curve.

If $A = C = 0$, $\frac{1}{\tau} = 0$, and the curve is a plane curve.
$K = \frac{A}{C} = \frac{0}{0}$. K, being indeterminate, a plane curve has an infinite number of Bertrand curves - they are the curves parallel to the given curve.

If $\rho = q_1$, $\tau = q_2$, in which q is constant the applica-

tion of (80) on page 29 of Eisenhart's Differential Geometry gives

$$x = \frac{q_1 q_2^2}{q_1^2 + q_2^2} \sin\left[\frac{\sqrt{q_1^2 + q_2^2}}{q_1 q_2} s\right],$$

$$y = -\frac{q_1 q_2^2}{q_1^2 + q_2^2} \cos\left[\frac{\sqrt{q_1^2 + q_2^2}}{q_1 q_2} s\right]$$

$$z = \frac{q_2 s}{\sqrt{q_1^2 + q_2^2}}.$$

This shows that the curve satisfying the condition $\rho = q_1 \quad \tau = q_2$ is a circular helix. In this case

$$\frac{A}{\rho} \text{ and } \frac{B}{\tau} = C,$$

becomes

$$\frac{A}{q_1} + \frac{B}{q_2} = C,$$

and

$$K = \frac{A}{C}$$

can be given any value. To each value of K corresponds a definite circular helix, having the same axis as the given helix. As the normal to a circular helix is perpendicular to the axis of the cylinder upon which the helix is traced, each value of K gives a different radius for the cylinder upon which each Bertrand curve is traced.

Where ρ and τ are functions of a parameter there will be in general only one value of A, B, and C satisfying the relation

$$\frac{A}{\rho} + \frac{B}{\tau} = C,$$

consequently, a twisted curve outside of the special cases noted, will have only one Bertrand curve, if, indeed, it has any.

V

CLASSES OF CURVES HAVING NO BERTRAND CURVES

The final problem of this paper is the investigation of curves whose coördinates are given as integral and rational algebraic functions of a parameter. In this problem $\frac{1}{\rho}$ and $\frac{1}{\tau}$ must be of the same degree if they are to satisfy the relation

$$\frac{A}{\rho} + \frac{B}{\tau} = C,$$

and consequently $\frac{1}{\rho^2}$ and $\frac{1}{\tau^2}$ will be of the same degree if the given curves are to have Bertrand curves.

First, let the curve be defined in functions of the same degree with the arc as the parameter.

$$x = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} \ . \ . \ . \ a_n$$

$$y = b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} \ . \ . \ . \ b_n$$

$$z = c_0 s^n + c\, s^{n-1} + c_2 s^{n-2} \ . \ . \ . \ c_n$$

Then $$\frac{1}{\rho^2} = x''^2 + y''^2 + z''^2$$

(1) $$= n^2(n-1)^2(a_0^2 + b_0^2 + c_0^2)s^{2n-4}$$
$$+ \ n(n-1)^2(n-2)(a_0a_1 + b_0b_1 + c_0c_1)s^{2n-5}$$
$$+ \ . \ . \ . \ . \ . \ . + 4\,(a_{n-2} + b_{n-2} + c_{n-2}).$$

The torsion is given by the formula

$$\frac{1}{\tau} = -\rho^2 \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}$$

Substituting the values of x' etcetera in the expression, one row of the determinant is

$$\begin{bmatrix} n\cdot a_0 s^{n-1} + (n-1)a_1 s^{n-2} + (n-2)a_2 s^{n-3} + \ldots + a_{n-1} \\ n(n-1)a_0 s^{n-2} + (n-1)(n-2)a\ s^{n-3} + (n-2)(n-3)a\ s^{n-4} + \ldots + 2a_{n-2} \\ n(n-1)(n-2)a_0 s^{n-3} + (n-1)(n-2)(n-3)a_1 s^{n-4} + \ldots + 6a_{n-3} \end{bmatrix}$$

The other rows differ only by having a_1 replaced by b_1 and c_1 respectively.

This determinant can be reduced to one in which one row is

$$\begin{bmatrix} 2(n-2)a_2 s^{n-3} + \ldots + (n-1)(n-2)a_{n-1} \\ (n-1)(n-2)a_1 s^{n-3} + \ldots + 2(n-2)a_{n-2} \\ n(n-1)(n-2)a_0 s^{n-3} + \ldots + 6a_{n-3} \end{bmatrix}$$

by multiplying row one by $(n-1)$, row two by s and subtracting, then row two by $(n-2)$, row three by s and subtracting, and finally, row one by $(n-2)$, row two by s and subtracting.

The determinant just found can be broken up into a number of determinants of which the one of highest degree in s is

$$\begin{vmatrix} 2(n-2)a_2 s^{n-3} & 2(n-2)b_2 s^{n-3} & 2(n-2)c_2 s^{n-3} \\ (n-1)(n-2)a\ s^{n-3} & (n-1)(n-2)b_1 s^{n-3} & (n-1)(n-2)c_1 s^{n-3} \\ n(n-1)(n-2)a\ s^{n-3} & n(n-1)(n-2)b_0 s^{n-3} & n(n-1)(n-2)c_0 s^{n-3} \end{vmatrix}$$

Consequently, the determinant in $\frac{1}{\tau}$ expands into a polynomial in s with the highest degree $(3n - 9)$, and $\frac{1}{\tau}$ may be written

$$\frac{1}{\tau^2} = \rho^4(R_0 s^{6n-18} + R\, s^{6n-19} + \dots + R_{6n-18})$$

$$= \frac{R_0 s^{6n-18} + R_1 s^{6n-19} + \dots + R_{n-18}}{Q_0 s^{4n-8} + Q_1 s^{4n-9} + \dots + Q_{4n-8}}$$

and $\frac{1}{\rho^2}$ is of the form

$$\frac{1}{\rho^2} = Q'_0 s^{2n-4} + Q'_1 s^{2n-5} + \dots + Q'_{2n-4}$$

Since $\frac{1}{\rho^2}$ and $\frac{1}{\tau^2}$ are of different degrees the given curve has no Bertrand curve.

Second, if the functions be of different degrees in s let then be arranged in the order

$$x = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n ,$$
$$y = b_r s^{n-1} + b_{r+1} s^{n-r-1} + \dots + b_n ,$$
$$z = c_{r+q} s^{n-r-q} + c_{r+q+1} s^{n-r-q-1} + \dots + c_n ,$$

$\frac{1}{\rho^2}$ may be found from the previous case by substitution of zeros, and because of the nature of the coefficients the degree of the expression will be unchanged and may be used as before,

$$\frac{1}{\rho^2} = Q'_0 s^{2n-4} + Q'_1 s^{2n-5} + \dots + Q'_{2n-4} .$$

Since the transformations of the determinant in $\frac{1}{\tau}$ will in general affect only the first column the final form will be a determinant in which the columns are,

$$\begin{bmatrix} A_1 s^{n-3} + A_2 s^{n-4} + \dots \\ A'_1 s^{n-3} + A'_2 s^{n-4} + \dots \\ A'' s^{n-3} + A'' s^{n-4} + \dots \end{bmatrix}$$

$$\begin{bmatrix} B_r s^{n-r-1} & + B_{r+1} s^{n-r-2} & + \dots & \\ 0 & + B_{r+1} s^{n-r-2} & + & \\ 0 & + \quad 0 & + B_{r+2} s^{n-r-3} + \dots \end{bmatrix}$$

$$\begin{bmatrix} C_{r+q} s^{n-r-q-1} & + C_{r+q+1} s^{n-r-q-2} + \dots & \\ 0 & + C'_{r+q+1} s^{n-q-r-2} + \dots & \\ 0 & + \quad 0 & + C''_{r+q+2} s^{n-r-q-3} + \dots \end{bmatrix}$$

The determinant of highest degree in s in this determinant is

$$\begin{vmatrix} A_1 s^{n-3} & B_{r+1} s^{n-r-2} & C_{r+q} s^{n-r-q-1} \\ A'_1 s^{n-3} & B'_{r+1} s^{n-r-2} & 0 \\ A''_1 s^{n-3} & 0 & 0 \end{vmatrix}$$

which is of degree $3n - 6 - 2r - q$.

$\frac{1}{T^2}$ is consequently a fraction with the degree of the numerator $6n - 12 - 4r - 2q$ and the degree of the denominator $4n - 8$. The case where $r = q = 0$ has been handled in the

previous example, and since in this case r and q cannot both be equal to zero $\frac{1}{\rho^2}$ and $\frac{1}{\tau^2}$ are of different degrees and consequently the curve has no Bertrand curve.

Finally, if the curve be given in a parameter which is not the arc the expressions for $\frac{1}{\rho^2}$ and $\frac{1}{\tau^2}$ must be taken

$$\frac{1}{\rho^2} = \frac{f_1''^2 + f_2''^2 + f_3''^2 - \varphi''^2}{\varphi''^2}$$

$$\frac{1}{\tau} = -\frac{\rho^2}{\varphi'^6}\begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_3''' & f_3''' \end{vmatrix},$$

where

$$\begin{aligned} x_1 &= f_1 = a_0u^n + a_1u^{n-1} + \dots + a_n \\ y &= f_2 = b_0u^n + b_1u^{n-1} + \dots + b_n \\ z &= f_3 = C_0u^n + C_1u^{n-1} + \dots + C_n \end{aligned}$$

and

$$\varphi' = \sqrt{f'^2 + f'^2 + f'^2}\,.$$

Therefore

$$\varphi'' = \frac{f_1'f_1'' + f_2'f_2'' + f_3'f_3''}{\sqrt{f'^2 + f'^2 + f'^2}}\,.$$

In this case $\frac{1}{\rho^2}$ is a fraction with the numerator of the degree $4n - 6$ and the denominator of the degree $6n - 6$. The determinant in $\frac{1}{\tau}$ by the previous method is of degree $3n - 9$, and $\frac{1}{\tau^2}$ will be a fraction with the numerator of the degree $8n - 30$ and the denominator of the degree $20n - 24$. As these expressions for $\frac{1}{\rho^2}$ and $\frac{1}{\tau^2}$ are not of the same degree no linear relation exists between them and the given curve has no Bertrand curve.

If x, y, and z are of different degrees with x of degree n, y of degree $n - r$, and z of degree $n - r - q$,

$\frac{1}{\rho^2}$ will still be of the same degree as previously, the determinant in $\frac{1}{\tau}$ by the previous method is of degree $3n - 6 - 2r - q$, and $\frac{1}{\tau^2}$ has a numerator of the degree $18n - 24 - 4r - 2q$ and the denominator of the degree $20n - 24$. As these expressions are not of the same degree, a final conclusion, that none of the curves under consideration have Bertrand curves, may be drawn.

APPENDIX

A

FROM THE FRENET-SERRET FORMULAS

$$
(1) \qquad
\begin{aligned}
\ell' &= -\left[\frac{\alpha}{\rho} + \frac{\lambda}{\tau}\right], \\
m' &= -\left[\frac{\beta}{\rho} + \frac{\mu}{\tau}\right], \\
n' &= -\left[\frac{\gamma}{\rho} \quad \frac{\nu}{\tau}\right],
\end{aligned}
$$

$$
(2) \qquad
\begin{aligned}
x' &= \alpha , \\
y' &= \beta , \\
z' &= \gamma .
\end{aligned}
$$

Differentiation of (2) and substitution of values from Frenet-Serret formulas gives

$$
(3) \qquad
\begin{aligned}
x'' &= \frac{\ell}{\rho} , \\
y'' &= \frac{m}{\rho} , \\
z'' &= \frac{n}{\rho} .
\end{aligned}
$$

Differentiation of (1) and substitution of values gives

$$(4)\qquad \begin{aligned} \ell'' &= -\left[\frac{\ell - \alpha\rho'}{\rho^2} + \frac{\ell - \lambda\tau'}{\tau^2}\right], \\ m'' &= -\left[\frac{m - \beta\rho'}{\rho^2} + \frac{m - \mu\tau'}{\tau^2}\right], \\ n'' &= -\left[\frac{n - \gamma\rho'}{\rho^2} + \frac{n - \gamma\tau'}{\tau^2}\right], \end{aligned}$$

where the notation follows Eisenhart's Differential Geometry, page 17. Equation (1), page 4, may be multiplied out and grouped in the form,

$$\begin{aligned} &\ell(y'z'' - y''z') + m(z'x'' - z''x') + n(x'y'' - x''y') \\ &+ K[\ell(n'x'' - n''x') + m(z'\ell'' - z''\ell') + n(x'm'' - x''m') \\ &+ \ell(m'z'' - m''z') + m(n'x'' - n''x') + n(\ell'y'' - \ell''y')] \\ &+ K^2[\ell(m'n'' - m''n') + m(n'\ell'' - n''\ell') + n(\ell'm'' - \ell''m')] = 0. \end{aligned}$$

Since the binomials in the first three terms are proportional to the direction cosines of the binormal and ℓ, m, and n are the direction cosines of the normal; the first three terms are equal to zero, because normal and binormal are perpendicular. The remaining expression may be divided by K as $K \neq 0$ which results in

$$(5)\qquad \begin{aligned} &\ell(y'n'' - y''n') + m(z'\ell'' - z''\ell') + n(x'm'' - x''m') \\ &+ \ell(m'z'' - m''z') + m(n'x'' - n''x') + n(\ell'y'' - \ell''y') \\ &+ K[\ell(m'n'' - n'm'') + m(n'\ell'' - n''\ell') + n(\ell'm'' - \ell''m')] = 0. \end{aligned}$$

The substitution of values found by (1), (2), (3), (4) in (5) reduces (5) to

$$\frac{\tau'}{\tau^2}(\ell\beta\nu + m\gamma\lambda + \alpha\mu n - \gamma\mu\ell - m\alpha\nu - \beta n\lambda)$$

(6) $$+ \frac{K\tau'}{\rho\tau^2}(\ell\gamma\mu - \ell\beta\nu + m\alpha\nu - m\gamma\lambda + n\beta\lambda - n\alpha\mu)$$

$$+ \frac{K\rho'}{\rho^2\tau}(\ell\nu\beta - \ell\mu\gamma + m\lambda\gamma - m\alpha\nu + n\mu\alpha - n\lambda\beta) = 0.$$

(6) may be grouped in the form

$$\frac{\tau'}{\tau}[\ell(\beta\nu - \gamma\mu) + m(\gamma\lambda - \alpha\nu) + n(\alpha\mu - \beta\lambda)]$$

(7) $$+ \frac{K\tau'}{\rho\tau^2}[\ell(\tau\mu - \beta\nu) + m(\alpha\nu - \gamma\lambda) + n(\beta\lambda - \alpha\mu)]$$

$$+ \frac{K\rho'}{\rho^2\tau}\,\ell(\beta\nu - \gamma\mu) + m(\gamma\lambda - \alpha\nu) + n(\alpha\mu - \beta\lambda)] = 0.$$

Since $\ell = \gamma\mu - \beta\nu$, $m = \alpha\nu - \gamma\lambda$, $n = \beta\lambda - \alpha\mu$ and $\ell^2 + m^2 = n^2 = 1$, (7) reduces to

(8) $$- \frac{\tau'}{\tau^2} + \frac{K\tau'}{\rho\tau^2} + \frac{K\rho'}{\rho^2\tau} = 0.$$

This is equation (2) of I from which (5) is obtained by integration.

APPENDIX

B

If $t^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$,

then

$$t\frac{\delta t}{ds} = (x - x_1)(\frac{\delta x}{ds} - \frac{\delta x_1}{ds}) + (y - y_1)(\frac{\delta y}{ds} - \frac{\delta y_1}{ds}) + (z - z_1)(\frac{\delta z}{ds} - \frac{\delta z_1}{ds}).$$

By substitution of the values found in equation (6),

$$t\frac{\delta t}{ds} = (x \div x\ )(\frac{dx}{ds} + 1 - \frac{y}{\rho} - \frac{dx_1}{ds} - 1 + \frac{y_1}{\rho})$$
$$+ (y - y\ )(\frac{dy}{ds} + \frac{x}{\rho} + \frac{z}{\tau} - \frac{dy_1}{ds} - \frac{x_1}{\rho} - \frac{z_1}{\tau})$$
$$+ (z - z\ )(\frac{dz}{ds} - \frac{y}{\tau} - \frac{dz_1}{ds} - \frac{y_1}{\tau}).$$

This reduces to

$$t\frac{\delta t}{ds} = (x - x_1)(\frac{dx}{ds} - \frac{dx_1}{ds}) + (y - y_1)(\frac{dy}{ds} - \frac{dy_1}{ds}) + (z - z_1)(\frac{dz}{ds} - \frac{dz_1}{ds})$$

that is;

$$t\frac{\delta t}{ds} = t\frac{dt}{ds}$$

or

$$\frac{\delta t}{ds} = \frac{dt}{ds}.$$

If $x_1 = x + at$

and by equation (6)

(1) $$\frac{\delta x}{ds} = \frac{dx}{ds} + 1 - \frac{y}{\rho},$$

then

$$\frac{\delta x}{ds} = \frac{\delta(x + at)}{ds} = \frac{\delta x}{ds} + \frac{\delta a}{ds} + \frac{\delta t}{ds}$$

by substitution of $\frac{\delta x}{ds}$,

$$\frac{\delta x_1}{ds} = \frac{dx}{ds} + 1 - \frac{y}{\rho} + t\frac{\delta a}{\quad} + a\frac{\delta t}{ds}\ .$$

By substitution of (1)

$$(2) \qquad \frac{dx_1}{ds} + 1 - \frac{y_1}{\rho} = \frac{dx}{ds} + 1 - \frac{y}{\rho} + t\frac{\delta a}{ds} + a\frac{\delta t}{ds}\ .$$

By differentiation of $x_1 = x + at$,

$$\frac{dx_1}{ds} = \frac{dx}{ds} + a\frac{dt}{ds} + t\frac{da}{ds}\ .$$

By substitution in (2)

$$\frac{dx}{ds} + a\frac{dt}{ds} + t\frac{da}{ds} + 1 - \frac{y_1}{\rho} = \frac{dx}{ds} + 1 - \frac{y}{\rho} + t\frac{\delta a}{ds} + a\frac{\delta t}{ds}$$

$$\frac{dt}{ds} = \frac{\delta t}{ds},$$

Therefore $$t\frac{\delta a}{ds} = \frac{da}{ds} + \frac{t(y - y_1)}{\rho\tau} = t\frac{da}{ds} - t\frac{b}{\rho}\ ,$$

or $$\frac{\delta a}{ds} = \frac{da}{ds} - \frac{b}{\rho}\ .$$

In the same way it may be shown that

$$\frac{\delta b}{ds} = \frac{db}{ds} + \frac{a}{\rho} + \frac{c}{\tau}\ ,$$

and $$\frac{\delta c}{ds} = \frac{dc}{ds} - \frac{b}{\tau}\ .$$

APPENDIX

C

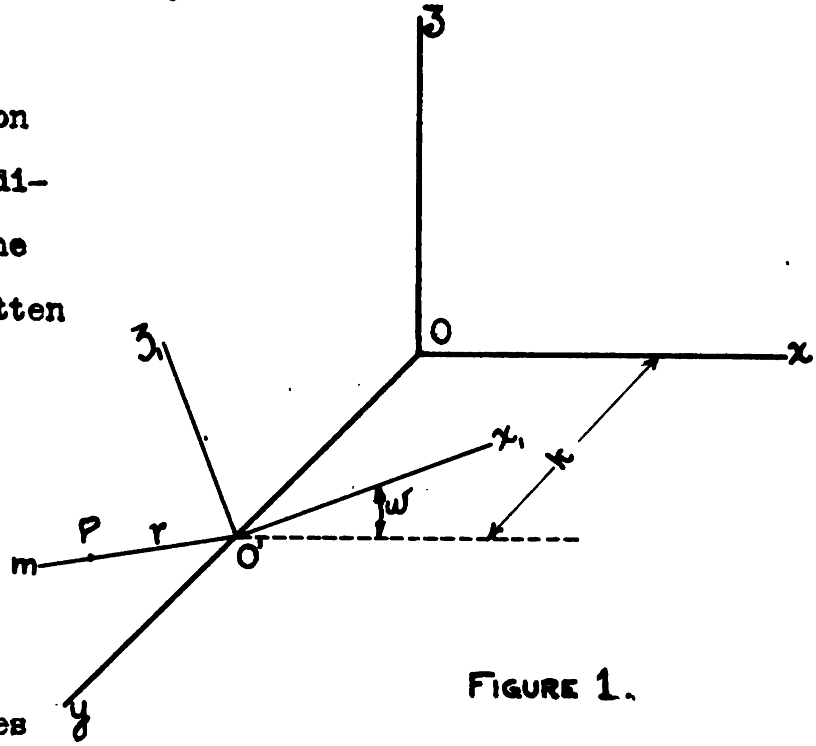

FIGURE 1.

If P is a point on the line mo' the coordinates with respect to the axis at o' may be written

$$x_1 = r\,a_1,$$

$$y_1 = r\,b_1,$$

$$z_1 = r\,C_1,$$

where a_1, b_1, and C_1, are the direction cosines of MO'.

Similarly with respect to the axes at o

$$x = ra\,,$$

$$y = rb + K,$$

$$z = rc\,.$$

The application of the formulas for translation and rotation of axes gives

$$ra = ra_1 \cos\omega + rc_1 \cos(90 + \omega),$$

$$rb + K = K + rb_1,$$

$$rc = ra_1 \cos(90 + \omega) + rc_1 \cos\omega,$$

These reduce to

$$a = a_1 \cos\omega - c_1 \sin\omega,$$

$$b = b\ ,$$

$$c = a_1 \sin\omega + c_1 \cos\omega,$$

Solving for a_1, b_1, c_1,

$$a_1 = a\cos\omega + c\sin\omega,$$

$$b_1 = b,$$

$$c_1 = a\sin\omega + c\cos\omega,$$

These are the relations given on page 13.

APPROVED: E. B. Skinner

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